Explicit Bounds for Generic Decoding Algorithms for Code-Based Cryptography

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1. Introduction

2. Asymptotic complexity of the Lee–Brickell algorithm

3. Asymptotic complexity of Stern’s algorithm
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2. Asymptotic complexity of the Lee–Brickell algorithm

3. Asymptotic complexity of Stern’s algorithm
Decoding problem

We only consider binary codes, i.e., codes over $\mathbb{F}_2$. In particular, we consider codes with no obvious structure.

**Classical decoding problem:** find the closest codeword $\mathbf{x} \in C$ to a given $\mathbf{y} \in \mathbb{F}_2^n$, assuming that there is a unique closest codeword.

Berlekamp, McEliece, van Tilborg (1978) showed that the general decoding problem for linear codes is NP-complete.

—p.1
McEliece PKC from an attacker’s point of view

Given a \( k \times n \) generator matrix \( G \) of a public code, and an error weight \( w \).

To encrypt a message \( m \in F_2^k \), the sender computes \( mG \), adds a random weight-\( w \) error vector \( e \), and sends \( y = mG + e \).

Not knowing the secret code and its decoding algorithm the attacker is faced with the problem of decoding \( y \) in a random-looking code.

McEliece proposed choosing random degree-\( t \) classical binary Goppa codes. The standard parameter choices are \( k = n - t \lceil \lg n \rceil \) and \( w = t \), typically with \( n \) a power of 2.

McEliece’s original suggestion: \( n = 1024, k = 524, \) and \( w = 50 \).
Attacks on the McEliece PKC

Most effective attack against the McEliece cryptosystem is information-set decoding.


Bernstein, Lange, P. (PQCrypto 2008): improved Stern attack and broke original McEliece parameters

Note: some of the algorithms are used for decoding; some are minimum-weight-word-finding algorithms. For comparison we rephrase all algorithms in terms of “fixed-distance decoding”.
Fixed-distance decoding

A fixed-distance-decoding algorithm searches for a codeword at a fixed distance from a received vector.

**Input:** the received vector $y$ and a generator matrix $G$ for the code.

**Output:** a sequence of weight-$w$ elements $e \in y - F_2^k G$.

Note that the output consists of error vectors $e$, rather than codewords $y - e$.

In the important special case $y = 0$, a fixed-distance-decoding algorithm searches for codewords of weight $w$. 
Information sets

Given a generator matrix $G$ of an $[n, k]$ code.

An information set is a size-$k$ subset $I \subseteq \{1, 2, \ldots, n\}$ such that the $I$-indexed columns of $G$ are invertible.

Denote the matrix formed by the $I$-indexed columns of $G$ by $G_I$. The $I$-indexed columns of $G_I^{-1}G$ are the $k \times k$ identity matrix.

Let $y \in \mathbb{F}_2^n$ have distance $w$ to a codeword in $\mathbb{F}_2^kG$, i.e., $y = c + e$ for a codeword $c \in \mathbb{F}_2^kG$ and a vector $e$ of weight $w$.

Denote the $I$-indexed positions of $y$ by $y_I$.

If $y_I$ is error-free, $y_I G_I^{-1}$ is the original message and $c = (y_I G_I^{-1})G$. Thus, $e = y - (y_I G_I^{-1})G$. 
1. Introduction

2. Asymptotic complexity of the Lee–Brickell algorithm

3. Asymptotic complexity of Stern’s algorithm
The Lee–Brickell algorithm

The algorithm consists of a series of independent iterations. Each iteration contains the following steps.

1. Select an information set $I \subseteq \{1, 2, \ldots, n\}$.
2. For each size-$p$ subset $A \subseteq \{1, \ldots, k\}$: compute

$$e = y - (y_IG_I^{-1})G - \sum_{a \in A} G_a,$$

where $G_a$ is the unique row of $G$ in which column $a$ has a 1; print $e$ if it has weight $w$.

A weight-$w$ error vector $e \in y - F_2^kG$ is found by an information set $I$ if and only if the $I$-indexed components of $e$ have weight $p$, and the remaining components of $e$ have weight $w - p$. 
Model of the number of iterations (Lee–Brickell)

If \( e \) is a uniform random weight-\( w \) element of \( \mathbb{F}_2^n \), and \( I \) is a size-\( k \) subset of \( \{1, \ldots, n\} \), then \( e \) has probability exactly

\[
LBPr(n, k, w, p) = \frac{\binom{n-k}{w-p} \binom{k}{p}}{\binom{n}{w}}
\]

of having weight exactly \( p \) on \( I \).

Consequently the Lee–Brickell algorithm, given \( c + e \) as input for some codeword \( c \), has probability exactly \( LBPr(n, k, w, p) \) of printing \( e \) in the first iteration.
The function $\text{LBCost}$ defined as

$$\text{LBCost}(n, k, w, p) = \frac{\frac{1}{2}(n - k)^2(n + k) + \binom{k}{p}p(n - k)}{\text{LBPr}(n, k, w, p)}.$$

is a model of the average time used by the Lee–Brickell algorithm.

- The term $\frac{1}{2}(n - k)^2(n + k)$ is a model of row-reduction time;
- $\binom{k}{p}$ is the number of size-$p$ subsets $A$ of $\{1, 2, \ldots, k\}$;
- and $p(n - k)$ is a model of the cost of computing $y - \sum_{a \in A} G_a$. 

—p.8
Asymptotic analysis

Let $R$ be the code rate and $S$ the error fraction $S$; i.e., $k = Rn$ and $w = Sn$.

**Goal:** Measure the scalability of the information-set algorithm.

The simplest form of information-set decoding takes time $2^{\alpha(R, S) + o(1)}n$ to find $Sn$ errors in a dimension-$Rn$ length-$n$ binary code if $R$ and $S$ are fixed while $n \to \infty$; here

$$\alpha(R, S) = (1 - R - S) \lg(1 - R - S) - (1 - R) \lg(1 - R) - (1 - S) \lg(1 - S)$$

and $\lg$ means the logarithm base 2.
Stirling revisited

We assume that the code rate $R = k/n$ and error fraction $S = w/n$ satisfy $0 < S < 1 - R < 1$.

We put bounds on binomial coefficients as follows. Define $\epsilon(m)$ for each integer $m \geq 1$ by the formula

$$m! = \sqrt{2\pi} m^{m+1/2} e^{-m+\epsilon(m)}.\]

The classic Stirling approximation is $\epsilon(m) \approx 0$. Robbins showed that

$$\frac{1}{12m + 1} < \epsilon(m) < \frac{1}{12m}. \tag{1}$$

Define $\text{LBErr}(n, k, w, p)$ as

$$\frac{k!}{(k-p)!k^p} \frac{w!}{(w-p)!w^p} \frac{(n-k-w)!(n-k-w)^p}{(n-k-w+p)!} e^{\epsilon(n-k)+\epsilon(n-w)} e^{\epsilon(n-k-w)+\epsilon(n)}.$$
Putting upper and lower bounds on $\text{LBPr}(n, k, w, p)$

Define $\beta(R, S) = \sqrt{(1 - R - S)/((1 - R)(1 - S))}$.

Lemma

$\text{LBPr}(n, k, w, p)$ equals

$$2^{-\alpha(R, S)n} \frac{1}{p!} \left( \frac{R Sn}{1 - R - S} \right)^p \frac{1}{\beta(R, S)} \text{LBErr}(n, k, w, p).$$

Furthermore

$$\frac{(1 - \frac{p}{k})^p (1 - \frac{p}{w})^p}{(1 + \frac{p}{n - k - w})^p} e^{-\frac{1}{12n} (1 + \frac{1}{1 - R - S})} < \text{LBErr}(n, k, w, p) < e^{\frac{1}{12n} (\frac{1}{1 - R} + \frac{1}{1 - S})}.$$

Note that for fixed rate $R$, fixed error fraction $S$, and fixed $p$ the error factor $\text{LBErr}(n, nR, nS, p)$ is close to 1 as $n$ tends to infinity.
Comparing Lee–Brickell for various $p$

**Corollary**

$LBCost(n, Rn, Sn, 0) = (c_0 + O(1/n))2^{α(R,S)n}n^3$ as $n → ∞$

where $c_0 = (1/2)(1 − R)(1 − R^2)β(R, S)$.

**Corollary**

$LBCost(n, Rn, Sn, 1) = (c_1 + O(1/n))2^{α(R,S)n}n^2$ as $n → ∞$

where $c_1 = (1/2)(1 − R)(1 − R^2)(1 − R − S)(1/RS)β(R, S)$.

**Corollary**

$LBCost(n, Rn, Sn, 2) = (c_2 + O(1/n))2^{α(R,S)n}n$ as $n → ∞$

where $c_2 = (1 − R)(1 + R^2)(1 − R − S)^2(1/RS)^2β(R, S)$.

**Corollary**

$LBCost(n, Rn, Sn, 3) = (c_3 + O(1/n))2^{α(R,S)n}n$ as $n → ∞$

where $c_3 = 3(1 − R)(1 − R − S)^3(1/S)^3β(R, S)$.
1. Introduction

2. Asymptotic complexity of the Lee–Brickell algorithm

3. Asymptotic complexity of Stern’s algorithm
Stern’s algorithm

Each iteration of Stern’s algorithm contains the following steps.

Fix parameters $\ell$ and $p$. Typically $\ell$ is chosen close to $\left(\frac{k}{2}\right)$.

1. Select an information set $I \subseteq \{1, 2, \ldots, n\}$.
2. Eliminate the $I$-indexed entries from $y$, i.e., replace $y$ by $y - y_I G^{-1}_I G$.
3. Select a uniform random size-$\left\lfloor \frac{k}{2} \right\rfloor$ subset $X \subseteq I$.
4. Define $Y = I \setminus X$.
5. Select a uniform random size-$\ell$ subset $Z \subseteq \{1, 2, \ldots, n\} \setminus I$.
6. For each size-$p$ subset $A \subseteq X$: Compute $\varphi(A) \in \mathbb{F}_2^\ell$, the $Z$-indexed entries of $y - \sum_{a \in A} G_a$.
7. For each size-$p$ subset $B \subseteq Y$: Compute $\psi(B) \in \mathbb{F}_2^\ell$, the $Z$-indexed entries of $\sum_{b \in B} G_b$.
8. For each pair $(A, B)$ such that $\varphi(A) = \psi(B)$: Compute $e = y - \sum_{a \in A} G_a - \sum_{b \in B} G_b$; print $e$ if it has weight $w$.

A weight-$w$ error vector $e \in y + \mathbb{F}_2^k G$ is found by an information set $I$ if and only if it has weight $p$ in the part corresponding to $X$, weight $p$ in the part corresponding to $Y$, and weight 0 in the part corr. to $Z$. —p.13
Model of the number of iterations (Stern)

If \( e \) is a uniform random weight-\( w \) element of \( \mathbb{F}_2^n \), \( I \) is a size-\( k \) subset of \( \{1, \ldots, n\} \), \( X \) is a size-\( k/2 \) subset of \( I \), and \( Z \) is a size-\( \ell \) subset of \( \{1, 2, \ldots, n\} \setminus I \), then \( e \) has probability exactly

\[
\text{STPr}(n, k, w, \ell, p) = \left( \frac{k}{2} \right)^2 \left( \frac{n - k - \ell}{w - 2p} \right) / \binom{n}{w}
\]

of having weights exactly \( p \) on \( X \), \( p \) on \( I \setminus X \), and 0 on \( Z \).

Consequently the Stern algorithm, given \( c + e \) as input for some code word \( c \), has probability exactly \( \text{STPr}(n, k, w, \ell, p) \) of printing \( e \) in the first iteration.
Model of the total cost

This function $\text{STCost}$ is a model of the average time used by Stern’s algorithm.

$$
\text{STCost}(n, k, w, \ell, p) = \frac{\frac{1}{2}(n-k)^2(n+k)+2\binom{k/2}{p}\ell + 2\binom{k/2}{p}^2 p(n-k)/2\ell}{\text{STPr}(n,k,w,\ell,p)}.
$$

- the term $\frac{1}{2}(n-k)^2(n+k)$ is a model of row-reduction time;
- $\binom{k/2}{p}$ is the number of size-$p$ subsets $A$ of $X$;
- $p\ell$ is a model of the cost of computing $\varphi(A)$; $\binom{k/2}{p}$ is the number of size-$p$ subsets $B$ of $Y$;
- $p\ell$ is a model of the cost of computing $\psi(B)$.

- we use $\binom{k/2}{p}^2/2^\ell$ as a model for the number of colliding pairs $(A, B)$.
- For each collision $2p(n-k)$ is a model of the cost of computing $y - \sum_{a \in A} G_a - \sum_{b \in B} G_b$. 

—p.15
Bounds on $\text{STPr}(n, k, w, \ell, p)$

Define error term as $\text{STErr}(n, k, w, \ell, p) = \frac{e^{\epsilon(n-k)+\epsilon(n-w)}}{e^{\epsilon(n-k-w)+\epsilon(n)}}
\cdot \left(\frac{(k/2)!}{(k/2-p)!(k/2)^p}\right)^2 \frac{w!}{(w-2p)!w^{2p}} \frac{(n-k-\ell)!(n-k)^\ell}{(n-k)!} \frac{(n-k-w)!}{(n-k-\ell-w+2p)!(n-k-w)^{\ell-2p}}.$

Lemma

$\text{STPr}(n, k, w, \ell, p)$ equals $2^{-\alpha(R,S)n} \frac{1}{(p!)^2} \left(\frac{RSV}{2(1-R-S)}\right)^{2p} \left(\frac{1-R-S}{1-R}\right)^\ell \frac{1}{\beta(R,S)} \text{STErr}(n, k, w, \ell, p).$

Furthermore

$(1 - \frac{2p}{k})^{2p} (1 - \frac{2p}{w})^{2p} (1 - \frac{n-k-\ell-w+2p}{n-k-w})p e^{-\frac{1}{12n}} \left(1 + \frac{1}{1-R-S}\right) < \text{STErr}(n, k, w, \ell, p) < (1 + \frac{\ell-1}{n-k-\ell-1})p e^{\frac{1}{12n}} \left(\frac{1}{1-R} + \frac{1}{1-S}\right).$
Most papers choose $\ell$ as $\lg \left( \frac{k}{2} \right)$ in order to balance $\left( \frac{k}{2} \right)$ with $\left( \frac{k}{2} \right)^2 / 2^\ell$.

However, starting from this balance, increasing $\ell$ by 1 produces better results: it chops $2\left( \frac{k}{2} \right)^2 p(n - k) / 2^\ell$ in half without seriously affecting $2\left( \frac{k}{2} \right) p\ell$ or $\text{STPr}(n, k, w, \ell, p)$.

Choosing $\ell$ close to $\lg \left( \frac{k}{2} \right) + \lg(n - k)$ would ensure that $2\left( \frac{k}{2} \right) p\ell$ dominates but would also significantly hurt $\text{STPr}$. 
Choice of parameters (2)

- (With any reasonable choice of $\ell$), increasing $p$ by 1 means that the dominating term $2\binom{k/2}{p}p\ell$ increases by a factor of approximately $k/(2p)$ while the denominator $\STPr(n, k, w, \ell, p)$ increases by a factor of approximately $(k/2p)^2w^2/(n - k - w)^2$.

- Overall $\STCost(n, k, w, \ell, p)$ decreases by a factor of approximately $(k/2p)w^2/(n - k - w)^2 = (R/2)(S/(1 - R - S))^2(n/p)$.

- The improvement from Lee–Brickell to Stern is therefore, for fixed $R$ and $S$, more than any constant power of $n$. 
There are several variants of information-set decoding designed to reduce the cost of row reduction, sometimes at the expense of success probability.

These variants save a non-constant factor for Lee–Brickell (LB) but save at most a factor $1 + o(1)$ for Stern. The critical point is that row reduction takes negligible time inside Stern’s algorithm, since $p$ is large.
Thank you for your attention!