Wild McEliece

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The end?

Quantum computers will break the most popular public-key cryptosystems:

- RSA,
- DSA,
- ECDSA,
- ECC,
- HECC
- ...

can be attacked in polynomial time using Shor's algorithm.

No! There's hope.

Post-quantum cryptography deals with public-key cryptosystems

- without (known) vulnerabilities to attacks by quantum computers and
- which run on conventional computers.

Examples are code-based cryptography, hash-based cryptography, lattice-based cryptography, and multivariate-quadratic-equations cryptography.

Overview:
Bernstein, Buchmann, and Dahmen, eds., Post-Quantum Cryptography. Springer, 2009.

1. Background on linear codes
2. The McEliece cryptosystem
3. Wild McEliece
4. Decoding Wild Goppa codes
5. Attacks
6. Parameters

Linear codes

A linear code C of length n and dimension k is a k-dimensional subspace of $\mathbf{F}_q^n$.

A generator matrix for C is a $k \times n$ matrix G such that $C = \left\{\mathbf{m}\, G : \mathbf{m} \in \mathbf{F}_q^k\right\}$.

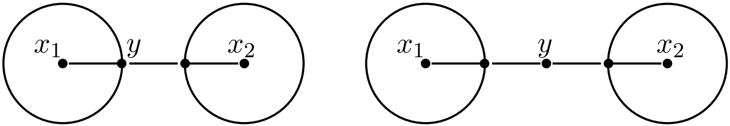
Example: The matrix

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

generates a code over $\mathbf{F}_2$ of length $n = 8$ and dimension $k = 4$.
Example of a codeword: $\mathbf{c} = (0110)G = (11111011)$.

Hamming distance

- The Hamming distance between two words in $\mathbf{F}_q^n$ is the number of coordinates where they differ.
- The Hamming weight of a word is the number of non-zero coordinates.
- The minimum distance of a linear code C is the smallest Hamming weight of a non-zero codeword in C.



code with minimum distance 3 code with minimum distance 4

Decoding problem

Classical decoding problem: find the closest codeword $\mathbf{c} \in C$ to a given $\mathbf{y} \in \mathbf{F}_q^n$, assuming that there is a unique closest codeword.

There are lots of code families with fast decoding algorithms

- E.g., Goppa codes/alternant codes, Reed-Solomon codes, Gabidulin codes, Reed-Muller codes, algebraic-geometric codes, BCH codes etc.

However, given a binary linear code with no obvious structure.

Berlekamp, McEliece, van Tilborg (1978) showed that the general decoding problem for linear codes is NP-complete.

- Best known attack: about $2^{(0.5+o(1))n/\log_2(n)}$ binary operations required for a code of length n and dimension $\approx 0.5n$.

Encryption

- Given public system parameters n, k, w.
- The public key is a random-looking $k \times n$ matrix $\hat{G}$ with entries in $\mathbf{F}_q$.
- Encrypt a message $m \in \mathbf{F}_q^k$ as

$$m\hat{G} + e$$

  where $e \in \mathbf{F}_q^n$ is a random error vector of weight w.
- Need to correct w errors to find m.
- Decoding is not easy without knowing the structure of the code generated by $\hat{G}$.

Secret key

The public key $\hat{G}$ has a hidden Goppa-code structure allowing fast decoding:

$$\hat{G} = SGP$$

where

- G is the generator matrix of a Goppa code Γ of length n and dimension k and error-correcting capability w;
- S is a random $k \times k$ invertible matrix; and
- P is a random $n \times n$ permutation matrix.

The triple (G, S, P) forms the secret key.

Note: Detecting this structure, i.e., finding G given $\hat{G}$, seems even more difficult than attacking a random $\hat{G}$.

Decryption

The legitimate receiver knows S, G and P with $\hat{G} = SGP$ and a decoding algorithm for Γ.

How to decrypt $y = m\hat{G} + e$.

1. Compute $yP^{-1} = mSG + eP^{-1}$.
2. Apply the decoding algorithm of Γ to find mSG which is a codeword in Γ from which one obtains m.

Goppa codes

- Fix a prime power q; a positive integer m, a positive integer $n \leq q^m$; an integer $t < \frac{n}{m}$;
- distinct elements $a_1, \ldots, a_n$ in $\mathbf{F}_{q^m}$;
- and a polynomial $g(x)$ in $\mathbf{F}_{q^m}[x]$ of degree t such that $g(a_i) \neq 0$ for all i.

The Goppa code $\Gamma_q(a_1, \ldots, a_n, g)$ consists of all words $c = (c_1, \ldots, c_n)$ in $\mathbf{F}_q^n$ with

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} \equiv 0 \pmod{g(x)}$$

- $\Gamma_q(a_1, \ldots, a_n, g)$ has length n and dimension $k \geq n - mt$.
- The minimum distance is at least $\deg g + 1 = t + 1$ (in the binary case $2t + 1$).

Reducing the key size (1)

- Binary Goppa code parameters achieving 128-bit security against the best known attack (Bernstein, Lange, P., PQCrypto 2008) produce a 1537536-bit key.

- Smaller-key variants use other codes such as Reed-Solomon codes, generalized Reed-Solomon codes, quasi-dyadic codes or geometric Goppa codes.

- Unfortunately, many proposals turned out to be breakable.

Reducing the key size (2)

- Goppa codes are the most confidence-inspiring choice.
- Using Goppa codes over larger fields decreases the key size at the same security level against information-set decoding (P., PQCrypto 2010).
- A Goppa code over $\mathbf{F}_{31}$ leads to a 725741-bit key for 128-bit security.
- Drawback: can correct only $\lfloor (t+1)/2 \rfloor$ errors if $q > 2$ (vs. t in the binary case).
- Today Wild Goppa codes.

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Proposal

Use the McEliece cryptosystem with Goppa codes of the form

$$\Gamma_q(a_1, \ldots, a_n, g^{q-1})$$

where g is an irreducible monic polynomial in $\mathbf{F}_{q^m}[x]$ of degree t.

- Note the exponent $q - 1$ in g^{q-1}.
- We refer to these codes as wild Goppa codes.

Minimum distance of wild Goppa codes

Theorem (Sugiyama-Kasahara-Hirasawa-Namekawa, 1976)

$$\Gamma_q(a_1, \ldots, a_n, g^{q-1}) = \Gamma_q(a_1, \ldots, a_n, g^q)$$

for a monic squarefree polynomial $g(x)$ in $\mathbf{F}_{q^m}[x]$ of degree t.

- The case $q = 2$ of this theorem is due to Goppa, using a different proof that can be found in many textbooks.

Proof

1. $\Gamma_q(a_1, \ldots, a_n, g^{q-1}) \supseteq \Gamma_q(a_1, \ldots, a_n, g^q)$:
   - If
     $$\sum_i \frac{c_i}{x - a_i} = 0 \text{ in } \mathbf{F}_{q^m}[x]/g^q$$
     then certainly
     $$\sum_i \frac{c_i}{x - a_i} = 0 \text{ in } \mathbf{F}_{q^m}[x]/g^{q-1}.$$

Proof (cont.)

2. $\Gamma_q(a_1, \ldots, a_n, g^{q-1}) \subseteq \Gamma_q(a_1, \ldots, a_n, g^q)$:
   - Consider any $(c_1, c_2, \ldots, c_n) \in \mathbf{F}_q^n$ such that $\sum_i c_i/(x - a_i) = 0$ in $\mathbf{F}_{q^m}[x]/g^{q-1}$.
   - Find an extension k of $\mathbf{F}_{q^m}$ so that g splits into linear factors in $k[x]$.
   - Then
   $$\sum_i \frac{c_i}{x - a_i} = 0 \text{ in } k[x]/g^{q-1},$$
   so
   $$\sum_i \frac{c_i}{x - a_i} = 0 \text{ in } k[x]/(x - r)^{q-1}$$
   for each factor $x - r$ of g.

Proof (cont.)

- The elementary series expansion

$$\frac{1}{x-a_i} = -\frac{1}{a_i-r} - \frac{x-r}{(a_i-r)^2} - \frac{(x-r)^2}{(a_i-r)^3} - \cdots$$

  then implies

$$\sum_i \frac{c_i}{a_i-r} + (x-r)\sum_i \frac{c_i}{(a_i-r)^2} + (x-r)^2 \sum_i \frac{c_i}{(a_i-r)^3} + \cdots = 0$$

  in $k[x]/(x-r)^{q-1}$.

- I.e., $\sum_i c_i/(a_i-r) = 0$,
  $\sum_i c_i/(a_i-r)^2 = 0$,
  $\ldots$,
  $\sum_i c_i/(a_i-r)^{q-1} = 0$.

Proof (cont.)

- Take the qth power of the equation $\sum_i c_i/(a_i - r) = 0$, to obtain $\sum_i c_i/(a_i - r)^q = 0$.
- Work backwards to see that $\sum_i c_i/(x - a_i) = 0$ in $k[x]/(x-r)^q$.
- By hypothesis g is the product of its distinct linear factors $x - r$.
- Therefore g^q is the product of the coprime polynomials $(x-r)^q$, and $\sum_i c_i/(x - a_i) = 0$ in $k[x]/g^q$.
- I.e.,
$$\sum_i \frac{c_i}{x - a_i} = 0 \text{ in } \mathbf{F}_{q^m}[x]/g^q.$$
- And thus $(c_1, \ldots, c_n) \in \Gamma_q(a_1, \ldots, a_n, g^q)$.

Error-correcting capability

- Since $\Gamma_q(\dots, g^{q-1}) = \Gamma_q(\dots, g^q)$ the minimum distance of $\Gamma_q(\dots, g^{q-1})$ equals the one of $\Gamma_q(\dots, g^q)$ and is thus $\geq \deg g^q + 1 = qt + 1$.

- We present an alternant decoder that allows efficient correction of $\lfloor qt/2 \rfloor$ errors for $\Gamma_q(\dots, g^{q-1})$.

- Note that the number of efficiently decodable errors increases by a factor of $q/(q-1)$ while the dimension $n - m(q-1)t$ of $\Gamma_q(\dots, g^{q-1})$ stays the same.

Sidestep: Number fields

- Consider the ring of integers $\mathcal{O}_L$ of a number field L and $Q_1, Q_2, \ldots$, the distinct maximal ideals of $\mathcal{O}_L$.

- A prime p ramifies in a number field L if the unique factorization $p\mathcal{O}_L = Q_1^{e_1} Q_2^{e_2} \cdots$ has an exponent e_i larger than 1.

- Each Q_i with $e_i > 1$ is ramified over p; this ramification is wild if e_i is divisible by p.

The "wild" terminology

- If $\mathcal{O}_L/p$ is $\mathbf{F}_p[x]/f$ for f a monic polynomial in $\mathbf{F}_p[x]$. Then $Q_1, Q_2, \ldots$ correspond to the irreducible factors of f, and $e_1, e_2, \ldots$ to the exponents in the factorization of f.
- The ramification corresponding to an irreducible factor ϕ of f is wild if and only if the exponent is divisible by p.
- We also refer to φ^p as being wild, and refer to the corresponding Goppa codes as wild Goppa codes.
- The traditional concept of wild ramification is defined by the characteristic of the base field.
- We take the freedom to generalize the definition of wildness to use the size of $\mathbf{F}_q$ rather than just the characteristic of $\mathbf{F}_q$.

Polynomial description of Goppa codes

Recall that

$$\begin{aligned}\Gamma &= \Gamma_q(a_1,\ldots,a_n,g^q)\\ &\subseteq \Gamma_{q^m}(a_1,\ldots,a_n,g^q)\\ &= \left\{\left(\frac{f(a_1)}{h'(a_1)},\ldots,\frac{f(a_n)}{h'(a_n)}\right) : f \in g^q\mathbf{F}_{q^m}[x], \deg f < n\right\}\end{aligned}$$

where $h = (x-a_1)\cdots(x-a_n)$.

- View target codeword $c = (c_1,\ldots,c_n) \in \Gamma$ as a sequence

$$\left(\frac{f(a_1)}{h'(a_1)},\ldots,\frac{f(a_n)}{h'(a_n)}\right)$$

of function values, where f is a multiple of g^q of degree below n.

Classical decoding

Given y, a word of distance $\lfloor qt/2 \rfloor$ from our target codeword. Reconstruct c from $y = (y_1, \ldots, y_n)$ as follows:

- Interpolate

$$\frac{y_1 h'(a_1)}{g(a_1)^q}, \frac{y_2 h'(a_2)}{g(a_2)^q}, \ldots, \frac{y_n h'(a_n)}{g(a_n)^q}$$

  into a degree-n polynomial $\varphi \in \mathbf{F}_{q^m}[x]$.
- Compute the continued fraction of φ/h to degree $\lfloor qt/2 \rfloor$.
- Compute $f = (\varphi - v_0 h/v_1) g^q$.
- Compute $c = (f(a_1)/h'(a_1), \ldots, f(a_n)/h'(a_n))$.

This algorithm uses $n^{1+o(1)}$ operations in $\mathbf{F}_{q^m}$ using standard FFT-based subroutines.

- A Python script can be found on my website: `http://www.win.tue.nl/~cpeters/wild.html`

Decoders

- Can use any Reed-Solomon decoder to reconstruct f/g^q from the values $f(a_1)/g(a_1)^q, \ldots, f(a_n)/g(a_n)^q$ with $\lfloor qt/2 \rfloor$ errors.

- This is an illustration of the following sequence of standard transformations:

  Reed–Solomon decoder $\Rightarrow$ generalized Reed–Solomon decoder
  $\Rightarrow$ alternant decoder $\Rightarrow$ Goppa decoder.

- The resulting decoder corrects $\lfloor (\deg g)/2 \rfloor$ errors for general Goppa codes $\Gamma_q(a_1, \ldots, a_n, g)$.

- In particular, $\lfloor q(\deg g)/2 \rfloor$ errors for $\Gamma_q(a_1, \ldots, a_n, g^q)$; and so $\lfloor q(\deg g)/2 \rfloor$ errors for $\Gamma_q(a_1, \ldots, a_n, g^{q-1})$.

List decoding

- Using the Guruswami–Sudan list-decoding algorithm we can efficiently correct $n - \sqrt{n(n-qt)} > \lfloor qt/2 \rfloor$ errors in the function values $f(a_1)/g(a_1)^q, \ldots, f(a_n)/g(a_n)^q$.
- Not as fast as a classical decoder but still takes polynomial time.
- Consequently we can handle $n - \sqrt{n(n-qt)}$ errors in the wild Goppa code $\Gamma_q(a_1, \ldots, a_n, g^{q-1})$.

Note:

- This algorithm can produce several possible codewords c. No problem for CCA2-secure variants of the McEliece system (Kobara, Imai, PKC 2001).
- We do not claim that this algorithm is the fastest possible decoder. Bernstein (2008) obtains for $q = 2$ the same error-correcting capability using a more complicated Patterson-like algorithm.

Attacks on Wild McEliece

- The wild McEliece cryptosystem includes, as a special case, the original McEliece cryptosystem.

- A complete break of the wild McEliece cryptosystem would therefore imply a complete break of the original McEliece cryptosystem.

Polynomial-searching attacks

- There are approximately q^{mt}/t monic irreducible polynomials g of degree t in $\mathbf{F}_{q^m}[x]$, and therefore approximately q^{mt}/t choices of g^{q-1}.

- An attacker can try to guess the Goppa polynomial g^{q-1} and then apply Sendrier's "support-splitting algorithm" to compute a permutation-equivalent code using the set $\{a_1, \ldots, a_n\}$.

- The support-splitting algorithm takes $\{a_1, \ldots, a_n\}$ as an input along with g.

Defenses

The first defense is well known and appears to be strong:

- Keep q^{mt}/t extremely large, so that guessing g^{q-1} has negligible chance of success. Our recommended parameters have q^{mt}/t dropping as q grows.

The second defense is unusual (strength is unclear):

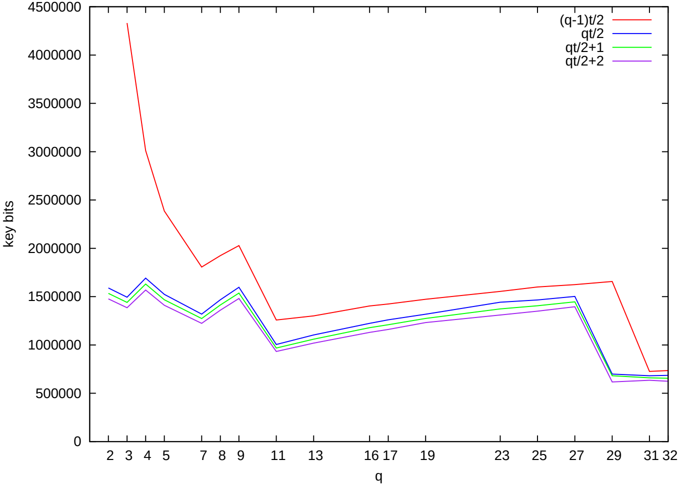
- It is traditional, although not universal, to take $n = 2^m$ and $q = 2$, so that the only possible set $\{a_1, \ldots, a_n\}$ is $\mathbf{F}_{2^m}$.
- Keep n noticeably lower than q^m, so that there are many possible subsets $\{a_1, \ldots, a_n\}$ of $\mathbf{F}_{q^m}$.
- Can the support-splitting idea be generalized to handle many sets $\{a_1, \ldots, a_n\}$ simultaneously?

Information-set decoding

- The top threat against the original McEliece cryptosystem is information-set decoding.
- The same attack also appears to be the top threat against the wild McEliece cryptosystem for $\mathbf{F}_3$, $\mathbf{F}_4$, etc.
- Use complexity analysis of state-of-the-art information-set decoding for linear codes over $\mathbf{F}_q$ from [P. 2010] to find parameters (q, n, k, t) for Wild McEliece.

Key sizes for various q at a 128-bit security level

McEliece with $\Gamma_q(a_1, \ldots, a_n, g^{q-1})$ and $\lfloor (q-1)t/2 \rfloor$, $\lfloor qt/2 \rfloor$, $\lfloor qt/2 \rfloor + 1$, or $\lfloor qt/2 \rfloor + 2$ added errors.



Thank you for your attention!