Wild McEliece Incognito

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joint work with
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1. Background

2. Wild McEliece Incognito

3. Attacks and defenses

4. Parameters

5. Challenges

Note

- This talk looks at “text-book” versions of cryptosystems.
- Plaintexts are not randomized.
- There exist CCA2-secure conversions of code-based cryptography which should be used when implementing the systems.

The McEliece cryptosystem

- Take a linear error-correcting code which can efficiently correct w errors.

- Publish a permutation-equivalent code and the error weight w.

- Encryption: sender embeds message into codeword in public code and adds w errors.

The well-known drawback

Key size determined by parameters of the secret code and its error correcting capability.

McEliece 1978: use as secret code a classical binary Goppa code $\Gamma_2(a_1, \ldots, a_n, g)$ (original parameters $n = 1024$, $w = \deg(g) = 50$).

Shrink keys: find a secret code

- allowing a compact representation
- such that the structure is not detectable from the public code given by some matrix;
- evaluate its security.

Attacking code-based cryptography

Two types of attacks in code-based cryptography:

- Generic decoding attacks: correct w errors in an arbitrary linear code.
- Structural attacks: try to find the secret code given the generator matrix of the public code.

Generic decoding attacks

Best known generic decoding attack relies on so-called information-set decoding.

Quite a long history:

1962 Prange; 1981 Clark (crediting Omura); 1988 Lee–Brickell; 1988 Leon; 1989 Krouk; 1989 Stern; 1989 Dumer; 1990 Coffey–Goodman; 1990 van Tilburg; 1991 Dumer; 1991 Coffey–Goodman–Farrell; 1993 Chabanne–Courteau; 1993 Chabaud; 1994 van Tilburg; 1994 Canteaut–Chabanne; 1998 Canteaut–Chabaud; 1998 Canteaut–Sendrier; 2008 Bernstein–Lange–P.; 2009 Finiasz–Sendrier; 2010 P.; 2011 Bernstein–Lange–P.; 2011 Sendrier; 2011 May–Meurer–Thomae.

Security

Suitable codes for code-based cryptography are such that the best attack is generic decoding.

- We say that a system has *b-bit security* if an attacker needs at least 2^b bit operations to decrypt a single ciphertext.

Structural attacks

- Generalized Reed–Solomon codes – broken by Sidelnikov and Shestakov (GRS variants broken by Wieschebrink)
- Gabidulin codes: broken by Gibson (variants broken by Overbeck)
- AG codes corresponding to GRS codes – broken by Minder, Minder–Faure, Pellikaan et al. (attacks generalize the Sidelnikov–Shestakov attack on GRS codes)
- quasi-cyclic Goppa codes and "non-binary" quasi-dyadic Goppa codes – broken (Gauthier–Leander, Faugère et al.)

Holding up

So far no structural attacks on code-based crypto using classical Goppa codes $\Gamma_q(a_1, \ldots, a_n, g)$.

- For given code parameters can build many different Goppa codes (thanks to subfield-subcode construction).

- Monoidic codes need further investigation.

Future work:

- Subfield subcodes of algebraic-geometric codes (avoid Sidelnikov–Shestakov-like attacks).

Way(s) to go

The more errors the secret code can correct — the harder the generic decoding problem.

- Yesterday's talk by Dan Bernstein: 1 extra error translates to a factor 5 in the complexity for generic attacks.

Two ways:

1. Improve decoding
   - Use list decoding for Goppa codes over $\mathbf{F}_2$ (see e.g., yesterday's talk by Dan Bernstein).
   - Better decoding algorithms for Goppa codes over bigger fields (yesterday's talk by Rafael Misoczki; improvement for $q = 3$).
2. Use subfamily of q-ary Goppa codes which can correct more errors with classical decoding algorithms.

Goppa codes

- Fix a prime power q; a positive integer m, a positive integer $n \leq q^m$; a positive integer t; distinct $a_1, \ldots, a_n \in \mathbf{F}_{q^m}$;

- and a polynomial $g(x)$ in $\mathbf{F}_{q^m}[x]$ of degree t such that $g(a_i) \neq 0$ for all i.

The Goppa code $\Gamma_q(a_1, \ldots, a_n, g)$ consists of all words $c = (c_1, \ldots, c_n)$ in $\mathbf{F}_q^n$ with

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} \equiv 0 \pmod{g(x)}$$

Properties of Goppa codes

- $\Gamma_q(a_1, \ldots, a_n, g)$ has length n and dimension $k \geq n - mt$.
- The minimum distance is at least $\deg g + 1 = t + 1$.
- However, a Goppa code over $\mathbf{F}_2$ has minimum distance at least $2t + 1$.
- Patterson decoding efficiently decodes t errors in the binary case whereas Berlekamp's algorithm corrects only $t/2$ errors for any q-ary Goppa code.

Proposal: Wild McEliece

Bernstein, Lange, P. at SAC 2010:

Use the McEliece cryptosystem with Goppa codes of the form

$$\Gamma_q(a_1, \ldots, a_n, g^{q-1})$$

where g is an irreducible monic polynomial in $\mathbf{F}_{q^m}[x]$ of degree t

- Note the exponent $q-1$ in g^{q-1}.
- We refer to these codes as wild Goppa codes.

Proposal: Wild McEliece Incognito

Beelen: hide wildness by using an extra factor.

This paper:

Use the McEliece cryptosystem with Goppa codes of the form

$$\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$$

where g is an irreducible monic polynomial in $\mathbf{F}_{q^m}[x]$ of degree t and f a irreducible monic polynomial coprime to g.

- Note the exponent $q-1$ in g^{q-1}.
- We refer to these codes as wild Goppa codes.

Features

- If $g = 1$ then $\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$ is the squarefree Goppa code $\Gamma_q(a_1, \ldots, a_n, f)$.

- If $f = 1$ then $\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$ is the *wild Goppa code* $\Gamma_q(a_1, \ldots, a_n, g^{q-1})$ (BLP10).

- The Goppa code with polynomial fg^{q-1} has dimension at least $n - m(s + (q-1)t)$, where s is the degree of f and t is the degree of g.

Minimum distance of wild Goppa codes

Theorem (Sugiyama–Kasahara–Hirasawa–Namekawa, 1976)

$$\Gamma_q(a_1, \dots, a_n, fg^{q-1}) = \Gamma_q(a_1, \dots, a_n, fg^q)$$

for coprime monic <u>*squarefree*</u> *polynomials* $g(x)$ *and* $f(x)$ *in* $\mathbf{F}_{q^m}[x]$ *of degree* t *and degree* s*, respectively.*

- The case $q = 2$ of this theorem is due to Goppa, using a different proof that can be found in many textbooks.

Error-correcting capability

- Since $\Gamma_q(\ldots, fg^{q-1}) = \Gamma_q(\ldots, fg^q)$ the minimum distance of $\Gamma_q(\ldots, fg^{q-1})$ equals the one of $\Gamma_q(\ldots, fg^q)$ and is thus $\geq \deg g^q + \deg f + 1 = qt + s + 1$.

- Alternant decoder efficiently corrects $\lfloor (qt+s)/2 \rfloor$ errors for $\Gamma_q(\ldots, fg^{q-1})$.

- Note that the number of efficiently decodable errors increases by up to a factor of $q/(q-1)$ while the dimension stays the same.

Decoding

- Can use any Reed–Solomon decoder to correct $\lfloor s + qt/2 \rfloor$ errors for $\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$.

Illustration of the following sequence of standard transformations:

$$\text{Reed–Solomon decoder} \Rightarrow \text{generalized Reed–Solomon decoder}$$
$$\Rightarrow \text{alternant decoder} \Rightarrow \text{Goppa decoder.}$$

In particular, can use list decoding:

- simplest case: Guruswami–Sudan list-decoding for Reed-Solomon codes.
- More sophisticated list-decoding algorithms can correct more errors and are faster.

Note: this talk will focus on deterministic decoding.

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Attacks on Wild McEliece

- The wild McEliece cryptosystem includes, as a special case, the original McEliece cryptosystem.

- A complete break of the wild McEliece cryptosystem would therefore imply a complete break of the original McEliece cryptosystem.

Polynomial-searching attacks

Case $\Gamma_q(a_1, \ldots, a_n, g^{q-1})$.

- There are approximately q^{mt}/t monic irreducible polynomials g of degree t in $\mathbf{F}_{q^m}[x]$, and therefore approximately q^{mt}/t choices of g^{q-1}.

- Knowing g^{q-1} an attacker can try to apply Sendrier's "support-splitting algorithm" to compute a permutation-equivalent code
  - requires knowledge of the support elements $\{a_1, \ldots, a_n\}$ in order to start the algorithm.

Defenses for the wild system

1. Make polynomial searching hard:
   - Keep q^{mt}/t extremely large, so that guessing g^{q-1} has negligible chance of success.

2. Give up traditional support length $n = q^m$:
   - Keep n noticeably lower than q^m, so that there are many possible subsets $\{a_1, \ldots, a_n\}$ of $\mathbf{F}_{q^m}$.
   - Can the support-splitting idea be generalized to handle many sets $\{a_1, \ldots, a_n\}$ simultaneously?

Incognito factor f

Completely avoid the potential problem of polynomial-searching attacks by using a code $\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$:

- Choose f so that finding fg^{q-1} takes at least as much time as generic decoding.
- For the extremely paranoid: choose parameters such that there are at least 2^{2b} possible polynomials fg^{q-1} when aiming at b-bit security against ISD.
- Note that factorizability of fg^{q-1} is not analogous to the concatenated structure attacked by Sendrier in 1994.

Masking the structure further

Add extra protection against structural attacks using an idea by Berger and Loidreau (2005).

- Add ℓ additional rows to parity-check matrix.
- There are $\binom{k}{\ell}_q = \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-\ell+1})}{(1-q)(1-q^2)\cdots(1-q^\ell)}$ subspaces of dimension ℓ in a k-dimensional code over $\mathbf{F}_q$ (already big for $\ell = 1$).
- Mask unsuccessful for GRS codes (Wieschebrink PQC2010); attack not obviously applicable to unmask Goppa codes.
- Small increase in key size: public key has $(n-k+\ell)(k-\ell)$ entries instead of $(n-k)k$ (systematic form).

Parameter choice

- The top threat against the original McEliece cryptosystem is information-set decoding.
- The same attack also appears to be the top threat against the wild McEliece cryptosystem for $\mathbf{F}_3$, $\mathbf{F}_4$, etc.
- Use complexity analysis of state-of-the-art information-set decoding for linear codes over $\mathbf{F}_q$ from [P. 2010] to find parameters (q, n, k, s, t) for Wild McEliece.

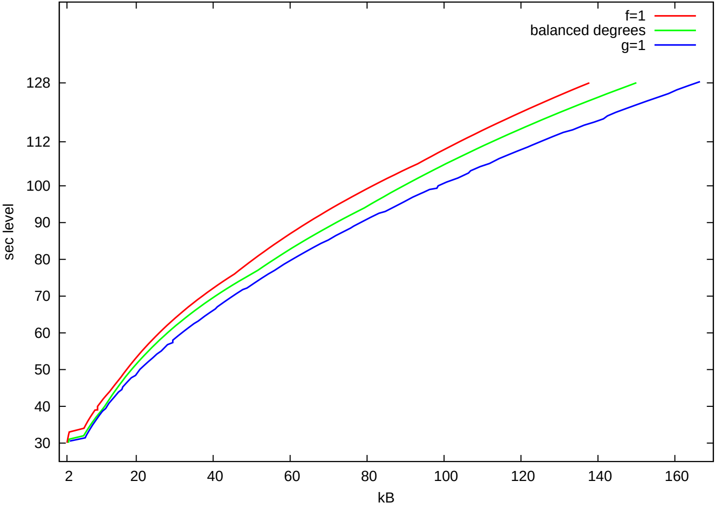
Parameter suggestions for 128-bit security

q	key size	n	k	s	t	w	p
3	186 kB	2136	1492	0	46	69	100%
4	210 kB	2252	1766	0	27	54	100%
5	191 kB	1878	1398	0	24	60	100%
7	170 kB	1602	1186	8	16	60	92%
8	187 kB	1628	1204	8	14	60	92%
9	205 kB	1668	1244	10	12	59	91%
11	129 kB	1272	951	17	9	58	84%
13	142 kB	1336	1033	17	7	54	83%
16	157 kB	1328	1010	16	6	56	85%
17	162 kB	1404	1113	17	5	51	82%
19	169 kB	1336	1015	17	5	56	84%
23	183 kB	1370	1058	16	4	54	85%
25	189 kB	1314	972	18	4	59	84%
27	200 kB	1500	1218	42	2	48	55%
29	199 kB	1390	1081	19	3	53	82%
31	88 kB	856	626	25	3	59	78%
32	89 kB	852	618	24	3	60	79%

Minimized key size against q-ary ISD attacks.

Key sizes for $q = 13$ for various security levels

McEliece with $\Gamma_q(a_1, \ldots, a_n, fg^{q-1})$ and $\lfloor (s + qt)/2 \rfloor$ added errors.



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How strong is the (wild) McEliece cryptosystem?

- Are there structural attacks against (wild) McEliece?
- How good/fast are the best generic attacks?
- In particular, measure progress of all sorts of attacks.
- How good are q-ary attacks?
  - Classical ISD vs. generalized statistical decoding (Niebuhr PQC2011).

Website (1)

Our challenges are online at

`http://pqcrypto.org/wild-challenges.html`

Each of our challenges is labelled by

1. "wild McEliece" for [BLP2010] or "wild McEliece incognito" for this paper;
2. a field size q ($q \geq 2$);
3. a key size expressed in kilobytes.

Website (2)

`http://pqcrypto.org/wild-challenges.html`

We intend to keep this web page up to date to show

- any solutions (plaintexts) sent to us — with credit to the first solver of each challenge;
- any secret keys sent to us — again with credit to the first solver of each challenge;
- cryptanalytic benchmarks — measurements of the speed of publicly available cryptanalytic software for the smaller challenges;
- predictions — estimates of how difficult the larger challenges will be to break.

Announcement

Code-based Cryptography
DTU, Lyngby
May 9–11, 2012

- Invited talks.
- Talks on recent results.
- Research retreat.

More information soon at

`http://www.agincc.mat.dtu.dk/`

Thank you for your attention!